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A novel and efficient analytical method for calculation of the transient temperature field in a multi-dimensional composite slab

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Abstract

This paper provides an efficient analytical tool for solving the heat conduction equation in a multi-dimensional composite slab subject to generally time-dependent boundary conditions. A temporal Laplace transformation and novel separation of variables are applied to the heat equation. The time-dependent boundary conditions are approximated with Fourier series. Taking advantage of the periodic properties of Fourier series, the corresponding analytical solution is obtained and expressed explicitly through employing variable transformations. For such conduction problems, nearly all the published works necessitate numerical work such as computing residues or searching for eigenvalues even for a one-dimensional composite slab. In this paper, the proposed method involves no numerical iteration. The final closed form solution is straightforward; hence, the physical parameters are clearly shown in the formula. The accuracy of the developed analytical method is demonstrated by comparison with numerical calculations.

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1. Introduction

Composite materials play an important role in today's industries. As a result, many engineering problems require a detailed knowledge of temperature distribution and heat flux in a composite slab. Examples exist in building physics, aerospace, thermodynamics, combustion, reacting flow processes, heat transfer, insulating technology, unconfined groundwater flows, and many others. Numerical methods are a common method of solving such problems; however, analytical methods can provide a greater insight into the physical process and can be used to validate numerical models. Among heat conductions in the composite slab, two- and three-dimensional conductions are the most important physical phenomena which need to be theoretically or experimentally studied. Such conduction problems are often simplified as extensions for one-dimensional geometry due to the highly complex nature of the multi-dimensional systems. Three analytical techniques, Green function, the orthogonal expansion and Laplace transform [1, 2] are often employed in tackling one-dimensional heat conduction. The same analytical techniques are also applied to multi-dimensional systems.

Regarding the first two techniques, examples are the works by Salt [3, 4], Mikhailov and Özisik [5] (the orthogonal expansion technique) and Beck [6] (the Green function). There are some limitations in applying these techniques. First, the technique of separation of variables often restricts boundary conditions. For example, the above-cited papers can only deal with the homogeneous boundary conditions of the first and second kind in the direction parallel to the layers. Second, associated eigenvalue problems are often inherited. Computations of eigenvalues for the multi-dimensional composite slab exhibit a few special features. The eigenvalues may become imaginary, so the corresponding eigenfuctions will have imaginary arguments [6]. Moreover, attention must be paid when computing eigenvalues since the spacing between successive eigenvalues varies between zero and the maximal value. Numerically, the imaginary eigenvalues can produce instability [6]. In fact, the associated eigenvalue problem is always complicated even for the one-dimensional composite slab [7]. With increasing layers, searching for eigenvalues may become so difficult as to be practically impossible.

de Monte made a detailed review on such techniques in solving multi-dimensional heat equations for the composite slab [8]. In that paper, the associated eigenvalue problem for two-dimensional composites with two rectangular parallel layers was split up into two one-dimensional eigenvalue problems. In the direction of the layers, the problem was a special case of the Sturm–Liouville problem. However, in the direction perpendicular to the layers, the problem was characterized by real and imaginary eigenvalues. Hence, the eigenfunctions across the layers were chosen with the target to identify the stress 'algebraic terms' which account for the heat conduction in the direction parallel to the layers and affect the thermal field in another direction [8].

Regarding the third technique, the Laplace transform, calculations often yield residue computation. For the composite slab, the computation is found by directly and numerically searching for the roots of a hyperbolic equation, finding the derivatives of the equation, and evaluating and summing the residues. The calculation procedure is tedious if the slab has more than two layers [9], as numerical searching roots have to be made with very fine increment for inverse Laplace transform to prevent missing roots which can lead to a wrong inverse.

Due to the above-discussed limitations, most of the published papers cannot deal with non-homogeneous boundary conditions for heat conduction problems. And they need to solve eigenvalue problems. The advantage over the numerical method is hard to discern. To improve the analytical methods, a novel analytical method was developed recently to tackle one-dimensional transient heat problems for a composite slab [10] and was later extended to multi-dimensional geometry by adopting a novel technique of separation of variables [11]. The boundary condition was presented as time-dependent temperature of the first kind.

In this paper, the developed method is extended for more general boundary conditions. The technique of separation of variables is used. The application of this technique, together with an employment of variable transformations, is novel which exhibits features other than the common ones, thus allowing for the complete range of boundary conditions to be explored including non-homogeneous boundaries in the direction parallel to the layers.

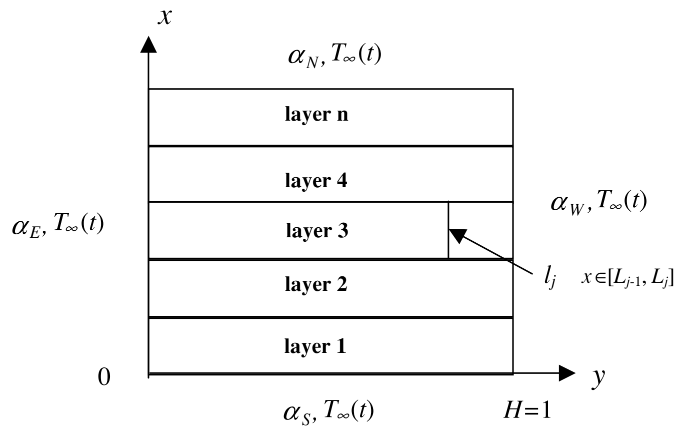

Figure 1. Schematic diagram of a two-dimensional composite slab.

The analytical approach in this paper presents a powerful method for solving multi-dimensional heat conduction equations in a composite slab.

Compared to the work reviewed above, firstly, the boundary condition is given more generally. Secondly, there is no need to numerically search for eigenvalues or to evaluate residues. Most importantly, explicit solutions for a multi-dimensional n-layer composite slab with general boundaries are available. The adopted method is efficient and straightforward. And the analytical solution is concise and easy to apply. Further a comparison of the results with numerical models demonstrates an application capability of the developed analytical method.

2. Mathematical formulation

2.1. Governing equations

Consider an n-layer composite slab having constant thermal conductivity, diffusivity and density in each layer. Each layer's thermal conductivity, diffusivity and thickness are presented as λ_j, k_j and l_j, $j = 1, \ldots, n$. The basic geometry considered here is a two-dimensional slab in x and y directions. The schematic figure is shown in figure 1.

So the layers have regional lengths $l_1, l_2, \ldots, l_n$. Denote $L_j = l_1 + \cdots + l_j$, $j = 1, \ldots, n$, then the layer boundaries are $x = L_0 = 0, L_1, \ldots, L_n$ and $y = 0, H$. For simplicity, it is assumed that $H = 1$.

The general heat conduction in the slab with the third kind boundary condition can be described by the following equation for temperature $T_j(t, x, y)$:

$$\frac{\partial T_j}{\partial t} = k_j \frac{\partial^2 T_j}{\partial x^2} + k_j \frac{\partial^2 T_j}{\partial y^2}, \qquad x \in [L_{j-1}, L_j], \quad y \in [0, 1], \quad j = 1, \ldots, n. \tag{2.1}$$

The boundary and initial conditions are

$$-\lambda_1 \frac{\partial T_1}{\partial x}(t, L_0, y) = -\alpha_S (T_1(t, L_0, y) - T_\infty(t)), \qquad y \in [0, 1], \tag{2.2a}$$

$$T_j(t, L_j, y) = T_{j+1}(t, L_j, y), \qquad y \in [0, 1], \quad j = 1, \ldots, n-1, \tag{2.2b}$$

$$-\lambda_j \frac{\partial T_j}{\partial x}(t, L_j, y) = -\lambda_{j+1} \frac{\partial T_{j+1}}{\partial x}(t, L_j, y), \qquad y \in [0, 1], \quad j = 1, \ldots, n-1, \tag{2.2c}$$

$$-\lambda_n \frac{\partial T_n}{\partial x}(t, L_n, y) = -\alpha_N(T_\infty(t) - T_n(t, L_n, y)), \qquad y \in [0, 1], \tag{2.2d}$$

$$-\lambda_j \frac{\partial T_j}{\partial y}(t, x, 0) = -\alpha_E(T_j(t, x, 0) - T_\infty(t)), \qquad x \in [L_{n-1}, L_n], \quad j = 1, \ldots, n, \tag{2.2e}$$

$$-\lambda_j \frac{\partial T_j}{\partial y}(t, x, 1) = -\alpha_W(T_\infty(t) - T_j(t, x, 1)), \qquad x \in [L_{n-1}, L_n], \quad j = 1, \ldots, n, \tag{2.2f}$$

$$T_j(0, x, y) = 0, \qquad x \in [L_{j-1}, L_j], \quad y \in [0, 1], \quad j = 1, \ldots, n. \tag{2.2g}$$

Without losing generality, the boundary temperature is expressed as $T_\infty(t) = \cos(\omega t + \varphi)$ and the initial temperature is assumed to be zero. The surface heat transfer coefficients are denoted as α with the subscripts representing different surfaces.

Even the homogeneous boundary condition of the third kind can produce mathematical incompatibilities in the direction parallel to the layers [8]; hence, only the first- and the second-kind boundary in the y-axis are considered. Therefore, α_E and α_W are assumed to be zero or ∞ (first and second kinds) which lead to four boundary conditions in the y-direction, namely:

$$y\text{-boundary-1:} \quad \alpha_E = \infty, \quad \alpha_W = \infty, \tag{2.3a}$$

$$y\text{-boundary-2:} \quad \alpha_E = \infty, \quad \alpha_W = 0, \tag{2.3b}$$

$$y\text{-boundary-3:} \quad \alpha_E = 0, \quad \alpha_W = 0, \tag{2.3c}$$

$$y\text{-boundary-4:} \quad \alpha_E = 0, \quad \alpha_W = \infty. \tag{2.3d}$$

2.2. Calculation procedures

We approach the analytical solutions by the following steps:

- Assume the boundary temperature as its complex form $T_\infty(t) = \mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}\varphi}$. Clearly, the solution of equations (2.1), (2.2) is the real part of the sought-after solution. If there is no danger of confusion we shall keep the same notations for the complex form of the boundary temperature.
- Reduce the two-dimensional problems into one-dimensional so that the available one-dimensional results can be applied. While the technique of separation of variables is a common method in tackling multi-dimensional homogeneous heat equations, this paper applied the technique in a new way accounting for the non-homogeneity of the boundaries.
- Mathematically, cases with y-boundary-2 and y-boundary-4 in equation (2.3) are the same. Therefore, we shall only consider three boundary conditions in the y-axis (y-boundary-1 to y-boundary-3). Closed form solutions will be provided for these boundary conditions.

3. Solutions to the first *y*-boundary condition

The first y-boundary condition assumes that $\alpha_E, \alpha_W = \infty$. Then the boundary condition in the y-axis becomes (see equations (2.2e)–(2.2f)):

$$T_j(t, x, 0) = T_\infty(t) = \mathrm{e}^{\mathrm{i}wt + \mathrm{i}\varphi}, \qquad x \in [L_{n-1}, L_n], \quad j = 1, \ldots, n, \tag{3.1a}$$

$$T_j(t, x, 1) = T_\infty(t) = \mathrm{e}^{\mathrm{i}wt + \mathrm{i}\varphi}, \qquad x \in [L_{n-1}, L_n], \quad j = 1, \ldots, n. \tag{3.1b}$$

In the following, we shall only write $x \in [L_{n-1}, L_n]$, $j = 1, \ldots, n$, if it is necessary.

3.1. Homogenizing the equations

The equation system has non-homogeneous boundary conditions. To homogenize, a new variable is introduced as

$$U_j(t, x, y) = T_j(t, x, y) - T_\infty(t). \tag{3.2}$$

This leads to the following non-homogeneous heat equation

$$\frac{\partial U_j}{\partial t} + T'_\infty(t) = k_j \frac{\partial^2 U_j}{\partial x^2} + k_j \frac{\partial^2 U_j}{\partial y^2}, \qquad x \in [L_{j-1}, L_j], \quad y \in [0, 1], \quad j = 1, \ldots, n. \tag{3.3}$$

with homogeneous boundary conditions

$$-\lambda_1 \frac{\partial U_1}{\partial x}(t, L_0, y) = -\alpha_S U_1(t, L_0, y), \qquad y \in [0, 1], \tag{3.4a}$$

$$U_j(t, L_j, y) = U_{j+1}(t, L_j, y), \qquad y \in [0, 1], \quad j = 1, \ldots, n-1, \tag{3.4b}$$

$$-\lambda_j \frac{\partial U_j}{\partial x}(t, L_j, y) = -\lambda_{j+1} \frac{\partial U_{j+1}}{\partial x}(t, L_j, y), \qquad y \in [0, 1], \quad j = 1, \ldots, n-1, \tag{3.4c}$$

$$-\lambda_n \frac{\partial U_n}{\partial x}(t, L_n, y) = \alpha_N U_n(t, L_n, y), \qquad y \in [0, 1], \tag{3.4d}$$

$$U_j(t, x, 0) = 0, \qquad x \in [L_{n-1}, L_n], \quad j = 1, \ldots, n, \tag{3.4e}$$

$$U_j(t, x, 1) = 0, \qquad x \in [L_{n-1}, L_n], \quad j = 1, \ldots, n, \tag{3.4f}$$

$$U_j(0, x, y) = -T_\infty(0), \qquad x \in [L_{j-1}, L_j], \quad y \in [0, 1], \quad j = 1, \ldots, n. \tag{3.4g}$$

3.2. Separating variables

Observing the common applications of separation of variables, the homogeneity of the system is needed so that the variables can be separated and the dimension can be reduced. Unfortunately, the heat equation here (equation (3.3)) does not have such a property. Therefore, in the following we shall separate the variables in an unusual way.

Assume 'separation of variables' can be used as

$$U_j(t, x, y) = X_j(t, x) Y_j(y), \tag{3.5}$$

where $Y_j(y)$ is a variable-separated function which satisfies the homogeneous form of equation (3.3). Then substituting $Y_j(y)$ into the homogeneous form of equation (3.3) results in

$$\text{function of } x \text{ and } t = \frac{k_j Y''_j}{Y_j}. \tag{3.6}$$

Setting each side of the above equation equal to $-\mu_j^2$ gives

$$Y''_j + \frac{\mu_j^2}{k_j} Y_j = 0. \tag{3.7}$$

The general solution Y_j is easily written as

$$Y_j(y) = A_j \sin\left(\frac{\mu_j}{\sqrt{k_j}} y\right) + B_j \cos\left(\frac{\mu_j}{\sqrt{k_j}} y\right). \tag{3.8}$$

Note that the boundary conditions (3.4*e*), (3.4*f*) are satisfied if $Y_j(0) = 0$ and $Y_j(1) = 0$ which lead to

$$\frac{\mu_j}{\sqrt{k_j}} = m\pi \quad \text{or} \quad \mu_{jm} = m\pi\sqrt{k_j} \qquad \text{and} \qquad Y_{jm}(y) = A_{jm}\sin\left(\frac{\mu_{jm}}{\sqrt{k_j}}y\right)$$
$$\text{or} \quad Y_m(y) = \sin(m\pi y), \qquad m = 1, \ldots, \infty. \tag{3.9}$$

The solution U_j in equation (3.5) can then be expressed as

$$U_j(t, x, y) = \sum_{m=1}^{\infty} X_{jm}(t, x)Y_m(y) = \sum_{m=1}^{\infty} X_{jm}(t, x)\sin(m\pi y). \tag{3.10}$$

Note that the unknown coefficient A_{jm} in equation (3.9) is embedded in X_{jm} in equation (3.10).

3.3. One-dimensional heat equation in *x*-variable

Unlike most of the conventional works, the corresponding heat equation for the x-variable cannot be obtained straightforward. We now give the details on how to approach the equation for x-variable. We shall omit writing $m = 1, \ldots, \infty$ if it cannot cause confusion.

Note that Y_m are orthogonal functions. Representing 1 as a sum of Y_m, the heat equation (3.3) is then obtained as

$$\sum_{m=1}^{\infty}\frac{\partial X_{jm}}{\partial t}Y_m + T'_\infty(t)\sum_{m=1}^{\infty} b_m Y_m = k_j\sum_{m=1}^{\infty}\frac{\partial^2 X_{jm}}{\partial x^2}Y_m - \sum_{m=1}^{\infty}\mu_{jm}^2 X_{jm}Y_m, \tag{3.11a}$$

where

$$b_m = \frac{2(1-\cos(m\pi))}{m\pi}. \tag{3.11b}$$

Simplification of the above equation (3.11*a*) gives

$$\frac{\partial X_{jm}}{\partial t} + b_m T'_\infty(t) = k_j\frac{\partial^2 X_{jm}}{\partial x^2} - \mu_{jm}^2 X_{jm}. \qquad x \in [L_{n-1}, L_n], \quad j = 1, \ldots, n. \tag{3.12}$$

Similarly, the boundary and initial conditions are derived from equations (3.4*a*)–(3.4*g*) as

$$-\lambda_1\frac{\partial X_{1m}}{\partial x}(t, L_0) = -\alpha_S X_{1m}(t, L_0) \tag{3.13a}$$

$$X_{jm}(t, L_j) = X_{(j+1)m}(t, L_j), \qquad j = 1, \ldots, n-1, \tag{3.13b}$$

$$-\lambda_j\frac{\partial X_{jm}}{\partial x}(t, L_j) = -\lambda_{j+1}\frac{\partial X_{(j+1)m}}{\partial x}(t, L_j), \qquad j = 1, \ldots, n-1, \tag{3.13c}$$

$$-\lambda_n\frac{\partial X_{nm}}{\partial x}(t, L_n) = \alpha_N X_{nm}(t, L_n), \tag{3.13d}$$

$$X_{jm}(0, x) = -b_m T_\infty(0), \qquad x \in [L_{n-1}, L_n], \quad j = 1, \ldots, n, \tag{3.13e}$$

where $T_\infty(t) = \mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}\varphi}$.

Note that equation (3.13*e*) is obtained by expressing 1 as a sum of Y_m in equation (3.4*g*). It can be observed that the derived one-dimensional heat equation for x-variable exhibits quite a different mathematical form than the original multi-dimensional equation does.

3.4. Analytical solution to one-dimensional equation in x-variable

3.4.1. Simplification of the equations. By introducing the new variable

$$V_{jm} = X_{jm} + \frac{\mathrm{i}\omega b_m}{\mu_{jm}^2 + \mathrm{i}\omega} T_\infty(t) + \frac{\mu_{jm}^2 b_m}{\mu_{jm}^2 + \mathrm{i}\omega} \mathrm{e}^{-\mu_{jm}^2 t + \mathrm{i}\varphi}, \tag{3.14}$$

equation (3.12) becomes

$$\frac{\partial V_{jm}}{\partial t} = k_j \frac{\partial^2 V_{jm}}{\partial x^2} - \mu_{jm}^2 V_{jm}, \qquad x \in [L_{n-1}, L_n], \quad j = 1, \ldots, n, \tag{3.15}$$

with boundary and initial conditions

$$-\lambda_1 \frac{\partial V_{1m}}{\partial x}(t, L_0) = -\alpha_S (V_{1m}(t, L_0) - G_{1m}), \tag{3.16a}$$

$$V_{jm}(t, L_j) = V_{(j+1)m}(t, L_j), \qquad j = 1, \ldots, n-1, \tag{3.16b}$$

$$-\lambda_j \frac{\partial V_{jm}}{\partial x}(t, L_j) = -\lambda_{j+1} \frac{\partial V_{(j+1)m}}{\partial x}(t, L_j), \qquad j = 1, \ldots, n-1, \tag{3.16c}$$

$$-\lambda_n \frac{\partial V_{nm}}{\partial x}(t, L_n) = \alpha_N (V_{nm}(t, L_n) - G_{nm}), \tag{3.16d}$$

$$V_{jm}(0, x) = 0, \qquad x \in [L_{n-1}, L_n], j = 1, \ldots, n, \tag{3.16e}$$

where

$$T_\infty(t) = \exp(\mathrm{i}\omega t + \mathrm{i}\varphi), \tag{3.16f}$$

$$G_{1m}(t) = \frac{\mathrm{i}\omega b_m}{\mu_{1m}^2 + \mathrm{i}\omega} T_\infty(t) + \frac{\mu_{1m}^2 b_m}{\mu_{1m}^2 + \mathrm{i}\omega} \mathrm{e}^{-\mu_{1m}^2 t + \mathrm{i}\varphi}, \tag{3.16g}$$

$$G_{nm}(t) = \frac{\mathrm{i}\omega b_m}{\mu_{nm}^2 + \mathrm{i}\omega} T_\infty(t) + \frac{\mu_{nm}^2 b_m}{\mu_{nm}^2 + \mathrm{i}\omega} \mathrm{e}^{-\mu_{nm}^2 t + \mathrm{i}\varphi}. \tag{3.16h}$$

3.4.2. Laplace transform of the equations. A very similar type of one-dimensional equations has been studied by Lu *et al* [10]; the implication of which is that the functions $G_{1m}(t)$ and $G_{nm}(t)$ may be considered as ambient temperature change. Lu *et al* suggested a method of Laplace transform on solving such equations.

Applying Laplace transform on equation (3.15) we get

$$s\bar{V}_{jm} = k_j \frac{\partial^2 \bar{V}_{jm}}{\partial x^2} - \mu_{jm}^2 \bar{V}_{jm}, \qquad x \in [L_{n-1}, L_n], \quad j = 1, \ldots, n \tag{3.17}$$

with boundary conditions

$$-\lambda_1 \frac{\partial \bar{V}_{1m}}{\partial x}(s, L_0) = -\alpha_S (\bar{V}_{1m}(s, L_0) - (s)), \tag{3.18a}$$

$$\bar{V}_{jm}(s, L_j) = \bar{V}_{(j+1)m}(s, L_j), \qquad j = 1, \ldots, n-1, \tag{3.18b}$$

$$-\lambda_j \frac{\partial \bar{V}_{jm}}{\partial x}(s, L_j) = -\lambda_{j+1} \frac{\partial \bar{V}_{(j+1)m}}{\partial x}(s, L_j), \qquad j = 1, \ldots, n-1, \tag{3.18c}$$

$$-\lambda_n \frac{\partial V_{nm}}{\partial x}(s, L_n) = \alpha_N (\bar{V}_{nm}(t, L_n) - \bar{G}_{nm}(s)). \tag{3.18d}$$

A bar over a function $f(t)$ designates its Laplace transform on t (e.g. [1]):

$$\bar{f}(s) = L(f(t)) = \int_0^{\infty} \exp(-s\tau) f(\tau)\, \mathrm{d}\tau. \tag{3.19a}$$

The Laplace transform of a convolution is given by

$$L(f_1(t) * f_2(t)) = \bar{f}_1(s)\bar{f}_2(s) \qquad \text{where} \quad f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau)\, \mathrm{d}\tau. \tag{3.19b}$$

3.4.3. Solutions to the equations. The general solution of equation (3.17) is obtained as

$$\bar{V}_{jm} = A_{jm} \sinh(q_{jm}(x - L_{j-1})) + B_{jm} \cosh(q_{jm}(x - L_{j-1})), \tag{3.20}$$

where A_{jm} and B_{jm} are determined with boundary conditions and

$$q_{jm} = \sqrt{\frac{s}{k_j} + \frac{\mu_{jm}^2}{k_j}} = \sqrt{\frac{s}{k_j} + m^2\pi^2}. \tag{3.21}$$

Setting $\xi_{jm} = q_{jm} l_j$ for $j = 1, \ldots, n$, and $h_j = \frac{\lambda_{j+1}}{\lambda_j}\sqrt{\frac{k_j}{k_{j+1}}}$ for $j = 1, \ldots, n-1$, the coefficients A_{jm} and B_{jm} in equation (3.20) are determined by the boundary conditions (3.18*a*)–(3.18*d*) as

$$\lambda_1 q_{1m} A_{1m} - \alpha_S B_{1m} = -\alpha_S \overline{G_{1m}} \tag{3.22a}$$

$$A_{jm} \sinh \xi_{jm} + B_{jm} \cosh \xi_{jm} - B_{(j+1)m} = 0, \qquad j = 1, \ldots, n-1, \tag{3.22b}$$

$$A_{jm} \cosh \xi_{jm} + B_{jm} \sinh \xi_{jm} - h_j A_{(j+1)m} = 0, \qquad j = 1, \ldots, n-1, \tag{3.22c}$$

$$A_{nm} h_{Am} + B_{nm} h_{Bm} = \alpha_N \overline{G_{nm}}, \tag{3.22d}$$

where

$$h_{Am} = \lambda_n q_{nm} \cosh \xi_{nm} + \alpha_N \sinh \xi_{nm}, \qquad h_{Bm} = \lambda_n q_{nm} \sinh \xi_{nm} + \alpha_N \cosh \xi_{nm}. \tag{3.22e}$$

The coefficients A_{jm} and B_{jm}, $j = 1, \ldots, n$, can be solved by Cramer's rule as follows: let

$$\Delta(s) = \begin{vmatrix} \lambda_1 q_{1m} & -\alpha_S & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \sinh \xi_{1m} & \cosh \xi_{1m} & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cosh \xi_{1m} & \sinh \xi_{1m} & -h_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \sinh \xi_{2m} & \cosh \xi_{2m} & 0 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh \xi_{2m} & \sinh \xi_{2m} & -h_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \sinh \xi_{(n-1)m} & \cosh \xi_{(n-1)m} & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cosh \xi_{(n-1)m} & \sinh \xi_{(n-1)m} & -h_{n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & h_{Am} & h_{Bm} \end{vmatrix}, \tag{3.23a}$$

$$\Delta_1(s) = \frac{\begin{vmatrix} \Delta(s) & \text{with} \\ \text{row} - 1 & \text{column} - 2j - 1 \\ \text{deleted} & \end{vmatrix}}{\Delta(s)}, \qquad \Delta_2(s) = \frac{\begin{vmatrix} \Delta(s) & \text{with} \\ \text{row} - 2n & \text{column} - 2j - 1 \\ \text{deleted} & \end{vmatrix}}{\Delta(s)}, \tag{3.23b}$$

$$\Delta_3(s) = \frac{\begin{vmatrix} \Delta(s) & \text{with} \\ \text{row} - 1 & \text{column} - 2j \\ \text{deleted} & \end{vmatrix}}{\Delta(s)}, \qquad \Delta_4(s) = \frac{\begin{vmatrix} \Delta(s) & \text{with} \\ \text{row} - 2n & \text{column} - 2j \\ \text{deleted} & \end{vmatrix}}{\Delta(s)}. \tag{3.23c}$$

Then

$$A_{jm} = -\alpha_S \overline{G_{1m}} \Delta_1 - \alpha_N \overline{G_{nm}} \Delta_2, \qquad B_{jm} = \alpha_S \overline{G_{1m}} \Delta_3 + \alpha_N \overline{G_{nm}} \Delta_4. \tag{3.24}$$

Without showing all calculation details, we give the closed form solution. More details can be found in Lu *et al* [10].

Put

$$M_{jm}(s, x) = -\alpha_S \Delta_1 \sinh(q_{jm}(x - L_{j-1})) + \alpha_S \Delta_3 \cosh(q_{jm}(x - L_{j-1})), \tag{3.25a}$$

$$N_{jm}(s, x) = -\alpha_N \Delta_2 \sinh(q_{jm}(x - L_{j-1})) + \alpha_N \Delta_4 \cosh(q_{jm}(x - L_{j-1})). \tag{3.25b}$$

Equation (3.20) is obtained as

$$\bar{V}_{jm} = M_{jm} \overline{G_{1m}} + N_{jm} \overline{G_{nm}}, \qquad x \in [L_{j-1}, L_j], \quad j = 1, \ldots, n. \tag{3.26}$$

The inverse function of $\bar{V}_{jm}$ can be approximated as

$$\begin{aligned} V_{jm}(t, x) = {} & \frac{\mathrm{i}\omega b_m}{\mu_{1m}^2 + \mathrm{i}\omega} M_{jm}(\mathrm{i}\omega, x)\, \mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}\varphi} + \frac{\mathrm{i}\omega b_m}{\mu_{nm}^2 + \mathrm{i}\omega} N_{jm}(\mathrm{i}\omega, x)\, \mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}\varphi} \\ & + \frac{\mu_{1m}^2 b_m}{\mu_{1m}^2 + \mathrm{i}\omega} M_{jm}\left(-\mu_{1m}^2, x\right) \mathrm{e}^{-\mu_{1m}^2 t + \mathrm{i}\varphi} + \frac{\mu_{nm}^2 b_m}{\mu_{nm}^2 + \mathrm{i}\omega} N_{jm}\left(-\mu_{nm}^2, x\right) \mathrm{e}^{-\mu_{nm}^2 t + \mathrm{i}\varphi}. \end{aligned} \tag{3.27}$$

3.4.4. Final solutions. From equations (3.2), (3.10), (3.14) and (3.27), we can simplify the solution as

$$\begin{aligned} X_{jm}(t, x) = {} & \frac{\mu_{1m}^2 b_m}{\mu_{1m}^2 + \mathrm{i}\omega} M_{jm}\left(-\mu_{1m}^2, x\right) \mathrm{e}^{-\mu_{1m}^2 t + \mathrm{i}\varphi} - \frac{\mu_{jm}^2 b_m}{\mu_{jm}^2 + \mathrm{i}\omega}\, \mathrm{e}^{-\mu_{jm}^2 t + \mathrm{i}\varphi} \\ & + \frac{\mu_{nm}^2 b_m}{\mu_{nm}^2 + \mathrm{i}\omega} N_{jm}\left(-\mu_{nm}^2, x\right) \mathrm{e}^{-\mu_{nm}^2 t + \mathrm{i}\varphi} + \left(\frac{\mathrm{i}\omega b_m}{\mu_{1m}^2 + \mathrm{i}\omega} M_{jm}(\mathrm{i}\omega, x) \right. \\ & \left. - \frac{\mathrm{i}\omega b_m}{\mu_{jm}^2 + \mathrm{i}\omega} + \frac{\mathrm{i}\omega b_m}{\mu_{nm}^2 + \mathrm{i}\omega} N_{jm}(\mathrm{i}\omega, x) \right) \mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}\varphi} \end{aligned} \tag{3.28}$$

and

$$T_j(t, x, y) = \mathbf{real} \left(\sum_{m=1}^{\infty} X_{jm}(t, x) \sin(m\pi y) \right) + \cos(\omega t + \varphi). \tag{3.29}$$

where **real** denotes the real part and $X_{jm}(t, x)$ is given in equation (3.28).

3.5. General boundary temperatures

For a more general boundary temperature $T_\infty(t)$, we approximate it as Fourier series $T_\infty(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(\omega_k t + \varphi_k)$. Due to the linear property of the equations, the solution is the sum of that with the constant boundary temperature a_0 and with the boundary temperature $\sum_{k=1}^{\infty} a_k \cos(\omega_k t + \varphi_k)$ which can be copied from the above-discussed theory.

For the constant boundary $T_\infty = a_0$, equation (3.26) is written as

$$\overline{V_{jm}} = \left(\frac{\mathrm{i}\omega b_m}{\mu_{1m}^2 + \mathrm{i}\omega} M_{jm} + \frac{\mathrm{i}\omega b_m}{\mu_{nm}^2 + \mathrm{i}\omega} N_{jm} \right) \frac{a_0}{s} + \cdots. \tag{3.30}$$

The omitted term presents the solutions with the periodic temperature boundary which is easily obtained from equation (3.29). As the matrix determinant M_{jm} or N_{jm} is the function of hyperbolic functions sinh and cosh which can be approximated by power series, a simple linearization of equation (3.30) gives

$$\overline{V_{jm}} \approx \frac{\text{const}}{\text{const}\,1 * s + \text{const}\,2}. \tag{3.31}$$

The inverse Laplace transform of the above equation is then

$$V_{jm} = \frac{\text{const}}{\text{const}\,1} \exp\left(-\frac{\text{const}\,2}{\text{const}\,1}t\right). \tag{3.22}$$

Hence, the final solution can be explicitly obtained.

Another simpler way of finding the solution for an equation with constant boundary change is ignoring the transient term which will die away if studies do not focus very much on the initial temperature change.

4. Solutions to the second *y*-boundary condition

The second y-boundary condition requires that $\alpha_E = \infty$, $\alpha_W = 0$. Then the boundary condition in the y-axis becomes (see equations (2.2e), (2.2f)):

$$T_j(t, x, 0) = T_\infty(t) = \mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}\varphi}, \qquad x \in [L_{n-1}, L_n], \quad j = 1, \ldots, n, \tag{4.1a}$$

$$\frac{\partial T_j}{\partial y}(t, x, 1) = 0. \qquad x \in [L_{n-1}, L_n], \quad j = 1, \ldots, n. \tag{4.1b}$$

The solution procedure can follow the theory developed for the first y-boundary condition. Hence we shall only present essential details in the following.

By introducing a new variable as equation (3.2) we reach the same equation system as equations (3.3), (3.4) except equation (3.4f) which is of the form

$$\frac{\partial U_j}{\partial y}(t, x, 1) = 0, \qquad x \in [L_{n-1}, L_n], \quad j = 1, \ldots, n, \tag{4.2}$$

Separating variables requires

$$\begin{aligned} &\frac{\mu_j}{\sqrt{k_j}} = \left(m + \tfrac{1}{2}\right)\pi \quad \text{or} \quad \mu_{jm} = \left(m + \tfrac{1}{2}\right)\pi\sqrt{k_j} \qquad \text{and} \\ &Y_{jm}(y) = A_{jm} \sin\left(\left(m + \tfrac{1}{2}\right)\pi y\right) = Y_m(y), \qquad m = 1, \ldots, \infty. \end{aligned} \tag{4.3}$$

The solution U_j can be written as

$$U_j(t, x, y) = \sum_{m=1}^{\infty} X_{jm} \sin\left(\left(m + \frac{1}{2}\right)\pi y\right). \tag{4.4}$$

Using the orthogonal property of Y_m to express 1 as a sum of Y_m, we get exactly the same equation (3.11a) for x-variable except that μ_{jm} and b_m are given in equation (4.3) and the following, respectively:

$$b_m = \frac{2\left(1 - \cos\left(m\pi + \frac{\pi}{2}\right)\right)}{(2m+1)\pi}, \qquad m = 1, \ldots, \infty. \tag{4.5}$$

Therefore, the closed form solution can be expressed as equations (3.28), (3.29) with the corresponding parameters μ_{jm} and b_m.

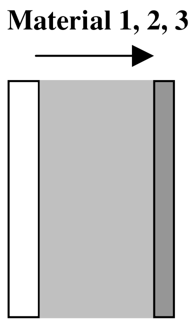

Figure 2. Schematic diagram of the three-layer composite slab.

5. Solutions to the third *y*-boundary condition

The third y-boundary condition requires that $\alpha_E = 0$, $\alpha_W = 0$. Then the boundary condition in the y-direction becomes (see equations (2.2e), (2.2f)):

$$\frac{\partial T_j}{\partial y}(t, x, 0) = 0, \qquad x \in [L_{n-1}, L_n], \quad j = 1, \ldots, n, \tag{5.1a}$$

$$\frac{\partial T_j}{\partial y}(t, x, 1) = 0, \qquad x \in [L_{n-1}, L_n], \quad j = 1, \ldots, n. \tag{5.1b}$$

Due to the adiabatic property in the y-axis, the equation system is reduced to the one-dimensional heat conduction in the composite slab which has been studied by Lu *et al* [10].

6. Calculation examples

6.1. Example description

This study has stemmed from a number of problems faced in our building physics research. At the moment, moisture is one of the primary causes of damage observed in building structures, increasing the importance of the development of research with the aim of finding regulations concerning the design of building walls with respect to moisture. One obvious concern is the control of the evolution of the moisture levels inside walls subject to the interior and exterior climatic conditions. For this purpose, we need to pursue analytical solutions so that investigation can take place into how the methodology described herein can be used to study equations such as the effects of parameter changes. The accuracy of the determination of the temperature distributions has an important effect on the final results of moisture calculations. Hence, we developed closed form analytical solution for heat conduction in composite slabs (building walls) as a starting point. Its accuracy is evaluated in this section by comparing the results with the numerical models.

A three-layer composite slab was selected as a calculation example. The composition has been used as the exterior wall in our test house. Figure 2 shows a schematic picture and table 1 illustrates the material properties of the composition.

The boundary temperature was taken from our laboratory measurement and fitted with periodic functions with periods 30, 5, 2 and 1 days as

$$T_\infty(t) = a_0 + \sum_1^4 a_i \cos\left(\frac{2\pi t}{\omega_i} - \varphi_i\right), \tag{6.1}$$

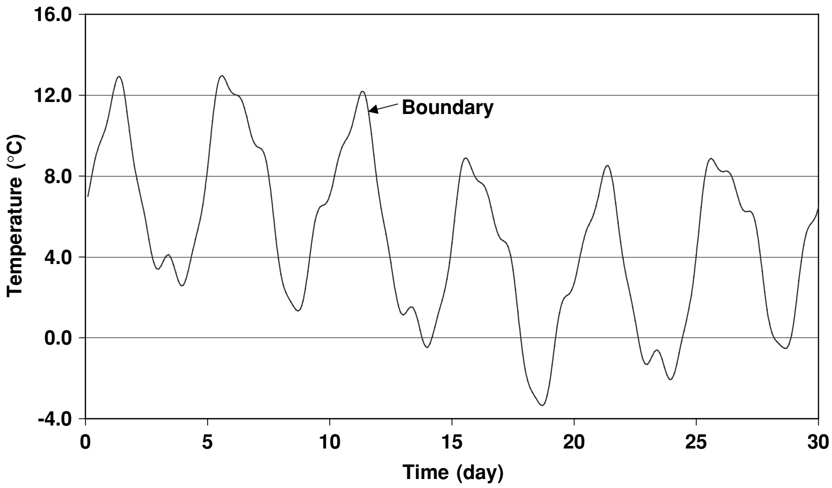

Figure 3. Boundary temperature changes.

Table 1. Material properties of the composite slab.

Material	Thermal conductivity ($\mathrm{W\,m^{-1}\,K^{-1}}$)	Thermal diffusivity ($\mathrm{m^2\,s^{-1}}$)	Thickness (mm)
1	0.12	1.5×10^{-7}	25
2	0.0337	1.47×10^{-6}	200
3	0.23	4.11×10^{-7}	13

Table 2. Parameters of equation (6.1).

	ω_1 30.0	ω_2 5.0	ω_3 2.0	ω_4 1.0
	φ_1 5.607 506	φ_2 13.595 96	φ_3 1.451 539	φ_4 5.418 717
a_0 5.0	a_1 2.722 17	a_2 $-5.019\,664$	a_3 1.084 058	a_4 0.4648

where fitting parameters are listed in table 2 and the graph is demonstrated in figure 3. Convective heat transfer coefficients were assumed to be $\alpha_N = 25\ \mathrm{W\,m^{-2}\,K^{-1}}$ and $\alpha_S = 6\ \mathrm{W\,m^{-2}\,K^{-1}}$.

6.2. Accuracy of the numerical model

As the analytical solution for the heat conduction in the three-layer composite is not available in general heat transfer contexts, we have to confine ourselves into a very simple case where the analytical solution is possible, for example a one-dimensional heat equation with first kind boundary. The comparison of a 100 terms truncation of the analytical solution with the numerical result shows a maximum error percentage of 0.2% [12]. The validation of the numerical model by experiments can be found in [13–15].

6.3. Comparison with the numerical model

Calculation was made at the central point of material 2. Figure 4 presents the comparison of the transient temperature variation using the analytical and numerical methods. The temperatures

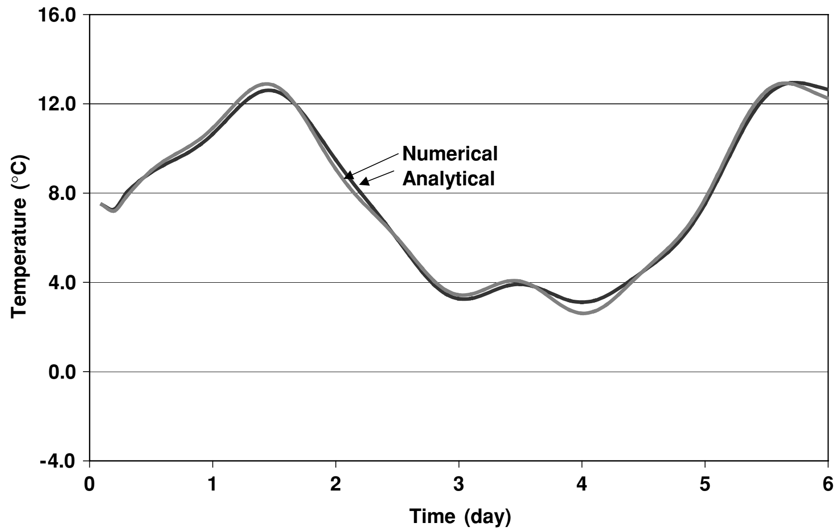

Figure 4. Comparison of the analytical and numerical results in the three-layer composite slab.

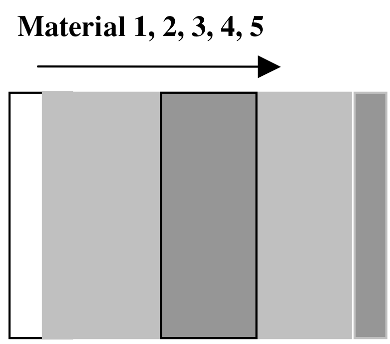

Figure 5. Schematic diagram of the five-layer composite slab.

were stored in files as hourly values and shown in figures as hourly and daily values. The maximal discrepancy is within 0.49 °C (relative error of 3%).

6.4. Another example

To demonstrate the capability of the analytical method, the three-layer composition in figure 2 is extended into the five-layer displayed in figure 5. The material properties are exhibited in table 3. A calculation was made at the centre of material 2. All other physical and environmental conditions were kept the same. The comparison results are shown in figure 6. The maximal discrepancy is within 1.8 °C with a relative error of 6%. The calculation example demonstrates a high accuracy of the developed solution.

6.5. Advantages over numerical models

For any *j*th layer of the composition, the attenuated temperature amplitude and time lag are able to be evaluated with the matrix determinants M and N in equation (3.25). Moreover, if

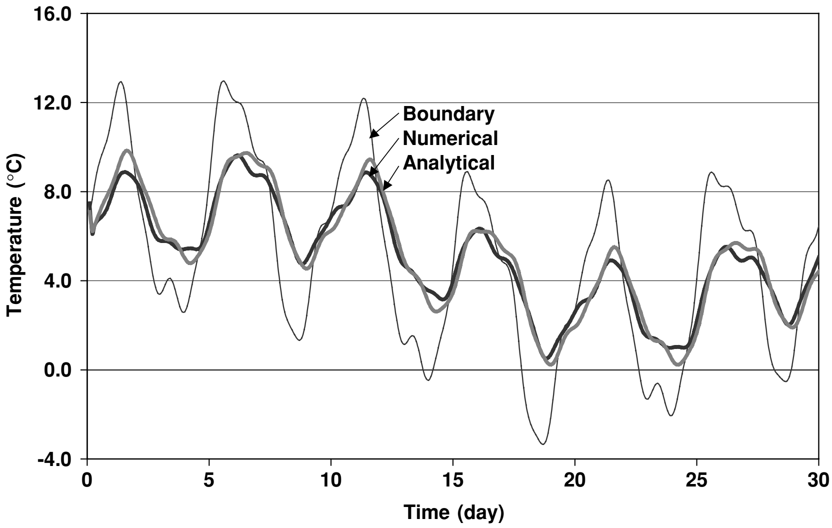

Figure 6. Comparison of the analytical and numerical results in the five-layer composite slab.

Table 3. Material properties of the composite slab.

Material	Thermal conductivity ($W\ m^{-1}\ K^{-1}$)	Thermal diffusivity ($m^2\ s^{-1}$)	Thickness (mm)
1	0.12	1.5×10^{-7}	50
2	0.0337	1.47×10^{-6}	200
3	0.9	3.75×10^{-7}	200
4	0.147	1.61×10^{-7}	200
5	0.23	4.11×10^{-7}	20

M and N are expressed algebraically as functions of material physical properties, for example thermal diffusivity, the effect on the solution of material physical parameters can be analysed. Such analysis cannot be performed with numerical models. Due to space constraints, we are not going to do such thermal analysis with the analytical model.

Moreover, the developed method is easier to implement and a possible instability in the numerical method is avoided. We have also made calculations on the three-dimensional case; no substantial change was found, yet the calculation load was much smaller and the computing time was much shorter compared with those when the numerical method was employed.

7. Conclusions

An analytical method was developed for solving multi-dimensional heat equations in the composite slab subject to generally time-dependent boundary conditions. This approach, based on the techniques of Laplace transform and separation of variables, made an approximation in calculating the inverse Laplace transformation and as a result a residue evaluation was avoided. The closed form solution formulae are obtained which are generally lacking in the literature due to the involvement of numerical procedures in searching eignevalues and residues. The solution formulae provide an insight into interplay between amplitude decays, time lags and other physical parameters, and can lead to better understanding of the thermal process in the composite slab. The range of applications is wide. Furthermore, agreement with numerical solutions is good. The method is easily extended to the three-dimensional composite slab without enduring any more computing load. In comparison with the numerical methods, the computing time is dramatically reduced especially in three-dimensional systems. Therefore, the analytical approach proves to be a powerful method for the study and the simulation of heat transfer phenomena in the composite slab.

It is worth mentioning that it is known that any periodic function can be represented as a Fourier series. A non-periodic function can be approximated as a Fourier series in the extended interval. For both the periodic and non-periodic functions, transient heat conduction must be solved during a short time interval compared with its period. Therefore, the solution results obtained in this paper are the transient solutions for general boundary conditions though the expressions are formulated with periodic functions, as demonstrated in the calculation example.

Finally, it is easy to observe that the calculation includes only simple computation of the matrix determinant. For any jth layer, only five sparse matrices are involved. The calculation load is small.

Acknowledgment

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